The Vasicek Interest Rate Process
Part V - Coupon Bond Price Equation

Gary Schurman, MBE, CFA

March, 2013

In Part III of the Vasicek Interest Rate Process series we calculated the price of a zero coupon, default free (i.e. no default risk) bond that was purchased at time s and matures at time t. In Part V we will calculate the price of a coupon paying, default free bond. To this end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with valuing a coupon paying, default free bond. The go-forward model assumptions are...

Description	Symbol	Value
Face value (in dollars)	F	1,000
Current short rate	r_s	0.04
Long-term short rate mean	r_∞	0.09
Contractual coupon rate	c	0.05
Annualized short rate volatility	σ	0.03
Mean reversion rate	λ	0.35

Question 1: Calculate the price of the coupon paying, default free bond above given that the bond was purchased at the end of year zero and matures at the end of year 10? Coupon payments are made every six months.

Question 2: How well does the approximation of the stochastic discount factor track the actual number?

The Stochastic Discount Factor

In Part II we defined the variable $R_{s,t}$ to be the random stochastic discount rate over the time interval $[s,t]$ and the variable r_u to be the random short rate at time u where $s \le u \le t$. The equation for the random stochastic discount rate is... [2]

$$R_{s,t} = \int_s^t r_u \, \delta u \tag{1}$$

We will define the variable Mean(s,t) to be the mean of the stochastic discount rate $R_{s,t}$ over the time interval $[s,t]$. In Part II we determined that the equation for the mean of the stochastic discount rate was... [2]

$$\text{Mean(s,t)} = r_\infty\,(t-s) + (r_\infty - r_s)\left(\text{Exp}\left\{-\lambda\,(t-s)\right\} - 1\right)\lambda^{-1} \tag{2}$$

We will define the variable Variance(s,t) to be the variance of the stochastic discount rate $R_{s,t}$ over the time interval $[s,t]$. In Part II we determined that the equation for the variance of the stochastic discount rate was... [2]

$$\text{Variance(s,t)} = \frac{\sigma^2}{2\,\lambda^3}\left(2\,\lambda\,(t-s) - 3 + 4\,\text{Exp}\left\{-\lambda\,(t-s)\right\} - \text{Exp}\left\{-2\,\lambda\,(t-s)\right\}\right) \tag{3}$$

In Equations (2) and (3) above r_s is the short rate at the beginning of the time interval $[s,t]$, r_∞ is the long-term short rate at time infinity, λ is the rate of mean reversion, and σ is the annual short rate volatility.

We will define the variable $\theta_{s,t}$ to be the stochastic discount factor over the time interval $[s,t]$. The stochastic discount factor can be viewed as follows...

$$\text{Value at time } s \text{ of } x \text{ dollars received at future time } t = x\,\theta_{s,t} \tag{4}$$

Using Equations (1), (2) and (3) above we will define the stochastic discount factor to be...

$$\theta_{s,t} = \mathbb{E}\Bigg[\mathrm{Exp}\Bigg\{-\int\limits_s^t r_u\,\delta u\Bigg\}\Bigg] = \mathrm{Exp}\Bigg\{-\,\text{Mean(s,t)} + \frac{1}{2}\,\text{Variance(s,t)}\Bigg\} \tag{5}$$

Bond Price Equation

In Part III we defined the variable $B(s,t)$ to be the price at time s of a zero coupon, default free bond that matures at time t and pays one dollar at maturity. The price of this bond may be written as the expectation of the path integral of the short rate over the time interval $[s,t]$. Using Equation (5) above the equation for bond price is... [3]

$$B(s,t) = \mathbb{E}\Bigg[\mathrm{Exp}\Bigg\{-\int\limits_s^t r_u\,\delta u\Bigg\}\Bigg] = \theta_{s,t} \tag{6}$$

We will define the variable $B(s,t,c)$ to be the price at time s of a coupon paying, default free bond that matures at time t and pays one dollar at maturity, the variable c to be the contractual coupon rate, and the variable α to be the continuous-time annual coupon rate. The price of this bond may be written as the discounted value of the principal repayment (the first half of the equation below) at time t plus the discounted value of coupon payments received over the time interval $[s,t]$ (the second half of the equation below). The equation for the price of this coupon paying bond is...

$$B(s,t,c) = \mathbb{E}\Bigg[\mathrm{Exp}\Bigg\{-\int\limits_s^t r_u\,\delta u\Bigg\}\Bigg] + \alpha\int\limits_s^t \mathbb{E}\Bigg[\mathrm{Exp}\Bigg\{-\int\limits_s^w r_u\,\delta u\Bigg\}\Bigg]\,\delta w = \theta_{s,t} + \alpha\int\limits_s^t \theta_{s,w}\,\delta w \tag{7}$$

We will define the variable n to be the number of coupon payments per annum. The equation to convert the contractual coupon rate to the continuous-time coupon rate is...

$$\alpha = \ln\left[\left(1+\frac{c}{n}\right)^n\right] \tag{8}$$

We will define the integral $I_{s,t}$ to be the following equation...

$$I_{s,t} = \int\limits_s^t \mathbb{E}\Bigg[\mathrm{Exp}\Bigg\{-\int\limits_s^w r_u\,\delta u\Bigg\}\Bigg]\,\delta w = \int\limits_s^t \theta_{s,w}\,\delta w = \int\limits_s^t \mathrm{Exp}\Bigg\{-\,\text{Mean(s,w)} + \frac{1}{2}\,\text{Variance(s,w)}\Bigg\}\,\delta w \tag{9}$$

Using Equation (9) we can rewrite coupon bond price Equation (7) above as...

$$B(s,t,c) = \theta_{s,t} + \alpha\,I_{s,t} \tag{10}$$

Given that solving integral $I_{s,t}$ above will be extremely difficult we will approximate that integral via an easy to integrate polynomial.

Approximating The Stochastic Disount Factor

We will define the variable $\hat{\theta}_{s,w}$ to be an approximation of the stochastic discount factor in Equation (5) above. The equation for $\hat{\theta}_{s,w}$ is the following polynomial...

$$\hat{\theta}_{s,w} = a\,w^2 + b\,w + c \tag{11}$$

Using the stochastic discount factor approximation in Equation (11) above we will rewrite Equation (9) above as...

$$I_{s,t} = \int\limits_s^t \hat{\theta}_{s,w}\,\delta w = \int\limits_s^t \Big(a\,w^2 + b\,w + c\Big)\delta w = a\int\limits_s^t w^2\,\delta w + b\int\limits_s^t w\,\delta w + c\int\limits_s^t \delta w \tag{12}$$

The solution to Equation (12)above is...

$$I_{s,t} = \frac{a}{3}\,w^3 \Bigg[_{w=s}^{w=t} + \frac{b}{2}\,w^2 \Bigg[_{w=s}^{w=t} + c\,w \Bigg[_{w=s}^{w=t} = \frac{1}{3}\,a\,(t^3 - s^3) + \frac{1}{2}\,b\,(t^2 - s^2) + c\,(t - s) \tag{13}$$

We need to estimate the variables a, b and c in Equation (11) above. Using Equation (5) above we will define the following matrices and vectors...

$$\mathbf{A} = \begin{bmatrix} t^2 & t & 1 \\ x^2 & x & 1 \\ s^2 & s & 1 \end{bmatrix} \text{ ...and... } \vec{\mathbf{v}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ ...and... } \vec{\mathbf{w}} = \begin{bmatrix} \theta_{s,w} \\ \theta_{s,x} \\ \theta_{s,s} \end{bmatrix} \text{ ...such that... } \mathbf{A}\,\vec{\mathbf{v}} = \vec{\mathbf{w}} \text{ ...where... } x = \frac{t+s}{2} \tag{14}$$

The solution to Equation (14) above (vector of variables a, b and c, which are the parameters to our polynomial approximation) is

$$\mathbf{A}^{-1}\,\vec{\mathbf{w}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{15}$$

Answers To Our Hypothetical Problem

Question 1: Calculate the price of the coupon paying, default free bond above given that the bond was purchased at the end of year zero and matures at the end of year 10? Coupon payments are made every six months.

Using the model parameters above and the short rate equation from Part I the expected short rate at the beginning of the time interval [3,7] is... [1]

$$r_3 = 0.09 + \text{Exp}\Bigg\{ - 0.35 \times 3 \Bigg\}\Bigg(0.04 - 0.09\Bigg) = 7.25\% \tag{16}$$

Using Equation (8)above and the model parameters above the continuous-time bond coupon rate is...

$$\alpha = \ln\left[\left(1 + \frac{0.05}{2}\right)^2\right] = 4.939\% \tag{17}$$

To approximate the pv of coupon payments we will construct a parabola as outlined in Equations (11) to (15) above. Using Equation (14) above as our guide the entries for **Matrix A** is...

Year	SDR Mean	SDR Variance	SDFactor	Notes
7	0.32234	0.00762	0.72722	SDR mean = Equation (2). Begin short rate = Equation (16)
5	0.15483	0.00147	0.85719	SDR variance = Equation (3)
3	0.00000	0.00000	1.00000	SD factor = Equation (5)

Using Equations (14) and (15) above and the data for **Matrix A** above is...

$$\text{if... } \mathbf{A} = \begin{bmatrix} 49 & 7 & 1 \\ 25 & 5 & 1 \\ 9 & 3 & 1 \end{bmatrix} \text{ ...and... } \vec{\mathbf{w}} = \begin{bmatrix} 0.72722 \\ 0.85719 \\ 1.00000 \end{bmatrix} \text{ ...then... } \vec{\mathbf{v}} = \mathbf{A}^{-1}\,\vec{\mathbf{w}} = \begin{bmatrix} 0.00161 \\ -0.08425 \\ 1.23831 \end{bmatrix} \tag{18}$$

Using Equation (13) above and the solution to **vector** $\vec{\mathbf{v}}$ above the approximation of the stochastic discount factor over the time interval $[3, 7]$ is...

$$I_{3,7} = \frac{1}{3} \times 0.00161 \times (7^3 - 3^3) + \frac{1}{2} \times 0.85719 \times (7^2 - 3^2) + 1.00000 \times (7 - 3) = 3.4373 \tag{19}$$

Using Equation (5) above and the data from **Matrix A** above the stochastic discount factor over the time interval $[3, 7]$ is...

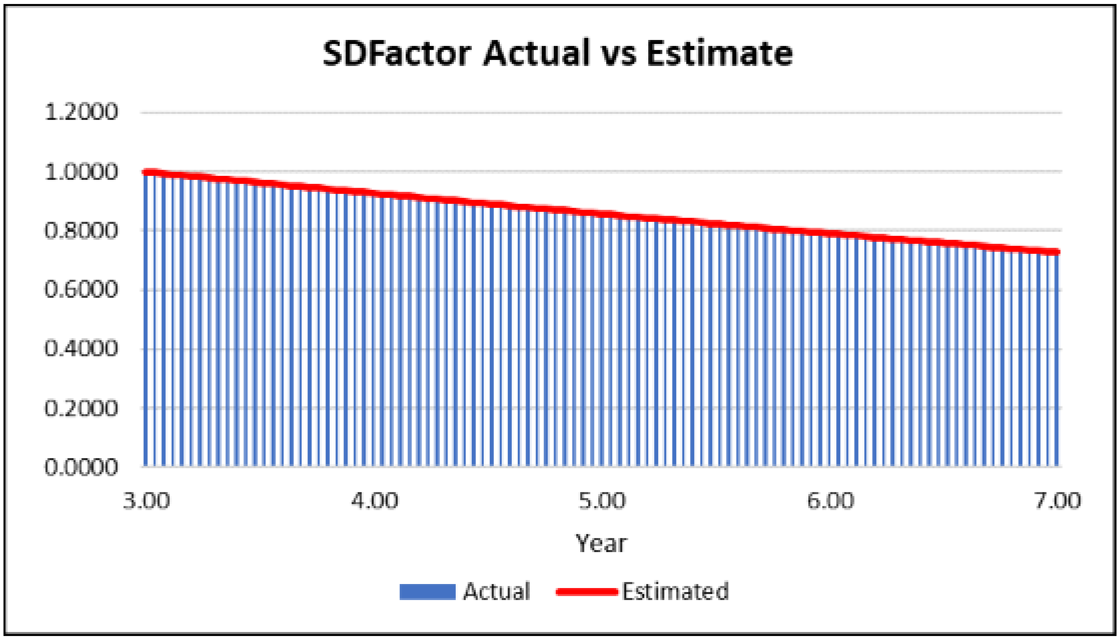
$$\theta_{3,7} = \text{Exp}\Bigg\{ - 0.32234 + \frac{1}{2} \times 0.00762 \Bigg\} = 0.72722 \tag{20}$$

Using bond price Equations (10), (17), (19) and (20) above to question one is...

$$\text{Bond Price} \;= F \times B(3,7,0.05) = F \times (\theta_{3,7} + \alpha\, I_{3,7}) = 1,000 \times (0.72722 + 0.04939 \times 3.4373) = 896.97 \tag{21}$$

Question 2: How well does the approximation of the stochastic discount factor track the actual number?

Per the following graph the stochastic discount factor approximated value tracks the actual value very well.



References

[1] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Short Rate,* February, 2013.

[2] Gary Schurman, *The Vasicek Interest Rate Process - The Stochastic Discount Rate,* February, 2013.

[3] Gary Schurman, *The Vasicek Interest Rate Process - Zero Coupon Bond Price,* March, 2013.